Exercise 3

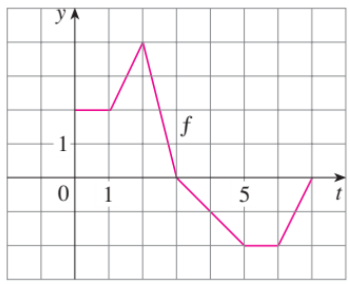
Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

(a) Evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$, and $g(6)$.

(b) On what interval is g increasing?

(c) Where does g have a maximum value?

(d) Sketch a rough graph of g.



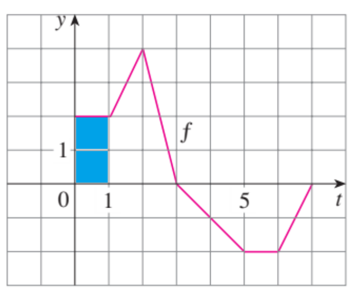
Solution

Part (a)

For $g(0)$, the limits of integration are the same, which makes the integral zero.

$$g(0) = \int_0^0 f(t)\,dt = 0$$

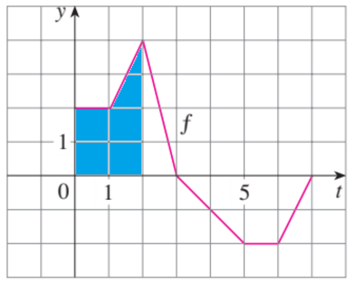
$g(1)$ is the area under the curve from $x = 0$ to $x = 1$.



The shape is a rectangle with base 1 and height 2.

$$g(1) = \int_0^1 f(t)\,dt = (1)(2) = 2$$

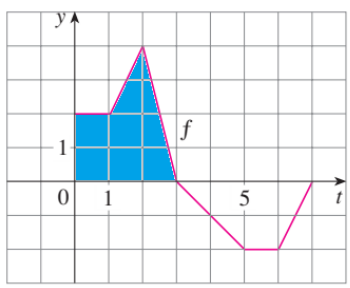
$g(2)$ is the area under the curve from $x = 0$ to $x = 2$.



The area under the curve from $x = 0$ to $x = 1$ is added to that from $x = 1$ to $x = 2$. The latter consists of a rectangle and a triangle—both with base 1 and height 2.

$$\begin{aligned} g(2) &= \int_0^2 f(t)\,dt \\ &= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \\ &= (1)(2) + \left[(1)(2) + \frac{1}{2}(1)(2)\right] \\ &= 5 \end{aligned}$$

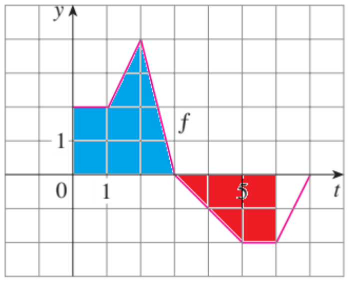
$g(3)$ is the area under the curve from $x = 0$ to $x = 3$.



The area under the curve from $x = 0$ to $x = 1$ is added to that from $x = 1$ to $x = 2$ and that from $x = 2$ to $x = 3$.

$$\begin{aligned} g(3) &= \int_0^3 f(t)\,dt \\ &= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt + \int_2^3 f(t)\,dt \\ &= (1)(2) + \left[(1)(2) + \frac{1}{2}(1)(2)\right] + \frac{1}{2}(1)(4) \\ &= 7 \end{aligned}$$

$g(6)$ is the area under the curve from $x = 0$ to $x = 6$.



The area under the curve from $x = 0$ to $x = 1$ is added to that from $x = 1$ to $x = 2$ and that from $x = 2$ to $x = 3$ and that from $x = 3$ to $x = 4$ and that from $x = 4$ to $x = 5$ and that from $x = 5$ to $x = 6$.

$$\begin{aligned} g(6) &= \int_0^6 f(t)\,dt \\ &= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt + \int_2^3 f(t)\,dt + \int_3^4 f(t)\,dt + \int_4^5 f(t)\,dt + \int_5^6 f(t)\,dt \\ &= (1)(2) + \left[(1)(2) + \frac{1}{2}(1)(2)\right] + \frac{1}{2}(1)(4) + \frac{1}{2}(1)(-1) + \left[(1)(-1) + \frac{1}{2}(1)(-1)\right] + (1)(-2) \\ &= 3 \end{aligned}$$

Part (b)

g is increasing on the interval $0 < x < 3$ because the curve is above the x-axis here.

Part (c)

The maximum of g occurs at $x = 3$: $g(3) = 7$.

Part (d)

Below is a plot of $g(x)$ versus x for $0 \leq x \leq 7$.